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Published in:

International Journal of Mathematical Education in Science and Technology

DOI:

[10.1080/0020739X.2019.1614688](https://doi.org/10.1080/0020739X.2019.1614688)

Publication date:

2020

Document version

Publisher's PDF, also known as Version of record

Citation for published version (APA):

Asami-johansson, Y., Attorps, I., & Winsløw, C. (2020). Comparing mathematics education lessons for primary school teachers: case studies from Japan, Finland and Sweden. *International Journal of Mathematical Education in Science and Technology*, 51(5), 688-712. <https://doi.org/10.1080/0020739X.2019.1614688>



International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: <https://www.tandfonline.com/loi/tmes20>

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To cite this article: Yukiko Asami-Johansson , Iiris Attorps & Carl Winsløw (2020) Comparing mathematics education lessons for primary school teachers: case studies from Japan, Finland and Sweden, International Journal of Mathematical Education in Science and Technology, 51:5, 688-712, DOI: [10.1080/0020739X.2019.1614688](https://doi.org/10.1080/0020739X.2019.1614688)

To link to this article: <https://doi.org/10.1080/0020739X.2019.1614688>



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Comparing mathematics education lessons for primary school teachers: case studies from Japan, Finland and Sweden

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ABSTRACT

The aim of this paper is to investigate and compare lessons given in primary school teacher education in Japan, Finland and Sweden. We analyse one lesson from each country and compare them using a common framework. Chevallard's anthropological theory of the didactic (ATD) is used to frame this analysis and in particular to model teacher educators' didactic organization of the lessons. The focus is on how the didactic organizations of the teacher educators relate to the mathematical and didactic organizations of primary school. Based on official documents and viewpoints of the teacher educators, we also discuss how the contents and descriptions of the national curricula, and the different traditions of the teaching practices in each country, influence the didactic organizations found in the lessons.

ARTICLE HISTORY

Received 2 October 2018

KEYWORDS

Teacher education; teaching knowledge; anthropological theory of the didactic; praxeology; mathematical organization; didactic organization; paradidactic infrastructure

1. Introduction

1.1. The use of comparative methods in research on teacher education

In the last few decades, international comparative studies in mathematics education, especially on classroom practices, have provided insight into differences in teaching cultures between Western and East Asian countries (e.g. [1–3]). Stigler and Perry [4] stress the importance, for researchers and educators, of cross cultural comparison for explicit understanding of pupils' learning of mathematics: 'Without comparison, we tend not to question our own traditional teaching practices and we may not even be aware of the choices we have made in constructing the educational process' [4,p.199]. Contributions and challenges of international comparisons of teacher education have appeared in recent years (e.g. [5]). Various cross-national studies have reported the main features of the mathematics teacher education in different countries (e.g. [6]), and a number of studies concern different aspects of student teachers' knowledge in pre-service teacher education (e.g. [7–9]).

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1.2. Studies regarding the provision of professional knowledge for prospective teachers

Considering providing the different kinds of professional knowledge for prospective teachers, several studies investigate the complexity of preparing them for the transition from being prospective teachers (hereafter, PTs) to becoming teachers (e.g. [10,11]). Winsløw et al. [12] have viewed the novice teachers' first years of teaching practice as a period of transition on mainly three interrelated levels: at an epistemological level: adapting their theoretical knowledge acquired in the pre-service education to the conditions of the practice of teaching; at an institutional level: passing from one institutional context (the university) to another (the school system); at a personal level: from being a student in a community of students to being a professional in a community of teachers (p.93).

The notion of the didactic divide is introduced by Bergsten and Grevholm [13] to illuminate the fundamental problem within teacher education in Sweden. They refer to Kilpatrick, Swafford and Findell [14] who state that teacher education needs to provide opportunities for PTs to connect different kinds of knowledge, and if certain connections are not realized, one may say there is a didactic divide between disciplinary and pedagogical knowledge¹ of mathematics. Bergsten and Grevholm also illustrate the situation in teacher education programmes, drawing on Ball and Bass's assertion that 'the teacher education across the twentieth century has consistently been severed by a persistent divide between subject matter knowledge and pedagogy', as 'the gap between subject matter knowledge and pedagogy fragments teacher education by fragmenting teaching' [15,p.85].

1.3. Teacher educators' teaching beliefs and teaching practices

Fewer studies concern teacher educators' teaching beliefs and teaching practices (e.g. Pope & Mewborn [16]). Concerning this issue, Hemmi and Ryve [17] made a comparison between Swedish and Finnish teacher educators' perception of 'effective mathematics teaching' by studying interview data with teacher educators (hereafter, TE/TEs) and school mentors. They reported that Swedish TEs tend to recommend PTs to adapt their teaching to individual pupils' thinking, and their everyday experiences. Finnish educators emphasize that mathematical teaching should connect to pupils' prior learned skills and should also balance the teaching focuses between routines, variation and homework. A comparative study conducted in Finland and Sweden [18] showed substantial differences of TEs' and teachers' views on the school-based teacher education between the countries.

1.4. Aim of this study

The aim of this paper is not to compare the teacher education programmes or teacher education contents in general. We compare how a certain subject is taught in what we can roughly call a methods course, that is a course on 'mathematics and its teaching' at the university – a kind of course which is distinct from both school-based teacher education and from normal mathematics courses; in many countries, such courses are meant as a kind of bridge between academic mathematics and teaching practice. As Liljedahl, Durand-Guerrier, Winsløw et al. [19] state, what is unique with teacher education is, 'what educators teach is also how educators teach, and what the prospective teachers learn is also

how they are learning' [19,p.29]. The task of teacher education is usually to make PTs learn the disciplinary knowledge in mathematics and, at the same time, its teaching approaches (thus, in fact, more than just mathematical knowledge and practice). Based on our review concerning primary mathematics teacher educators' practices, we found that there is little (if any) research on the contents and activities in methods courses focusing on this nested structure of didactic knowledge.

This study describes and compares the lessons in three different countries, to reveal crucial conditions and constraints forming the practices in methods courses at the primary school mathematics teacher education programmes. Our focus will be on the epistemological level [12] with the aim of investigating how the three TEs deal with the theoretical knowledge and practice of teaching mathematics. For a comparison to the Swedish context, we chose Finland which has significantly better results in mathematical literacy assessments [20]; and Japan, where the teaching culture in mathematics is reported to be more collective, compared to that of the US and Europe [21]. In this paper, we compare lessons from methods courses concerning the area of figures in the plane, given in the three aforementioned countries.

2. Theoretical framework and research questions

From the perspective of the anthropological theory of the didactic (ATD) developed by Chevallard and his colleagues, mathematics learning is considered as a construction of *praxeologies* [22] within social institutions where different levels of mathematical knowledge are required. A *praxeology* is a model of human activity and it provides both methods for the solution of a domain of problems (*praxis*) and a structure (*logos*) for the discourse on the methods and their relations to broader settings. Here, the *praxis* part consists of a *type of tasks* (T) and a *technique* (τ) to solve the (T), and the *logos* part includes a *technology* (θ), which explains and justifies the techniques, and a *theory* (Θ), which justifies and explains the technology more generally and formally. Two special kinds of praxeologies are also denoted, more specifically, *mathematical organizations* (MO) and *didactic organizations* (DO). A MO is a praxeology where the type of tasks is mathematical, and a DO is a praxeology where the type of tasks concerns the support of the learning or teaching of a MO. Thus those two kinds of organizations are mutually dependent or, as Chevallard [23] puts it, *co-determined*. The *praxis* part of the teacher educators (TEs)' DO is therefore consists of the types of didactic tasks (e.g. 'providing the prospective teachers (PTs) a certain teaching method of addition of 2-digits numbers'), and TEs' didactic techniques (e.g. 'giving PTs the task of writing a report regarding the teaching methods of addition'; 'letting the PTs demonstrate an example lesson during the class'). The use of the notion of praxeology make possible for researchers to recognize and categorize the actual components of the teaching and learning activities in every educational level (e.g. about analysis on the task including proportional relationships in elementary school level in Sweden [24]; Wijayanti & Winslow (2017), about analysis of the mathematical content in Indonesian textbooks in secondary school level [25]; about the difference of the type of tasks and associated techniques between university and secondary level [26]).

Our study adopts this tool to characterize TEs and PTs' activities in the teaching methods class, in order to make explicit what kinds of mathematical and didactic practice and knowledge are at stake there. From the viewpoint of the praxeology, the purpose of the

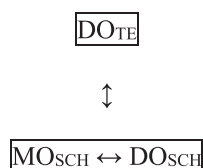


Figure 1. The complex of praxeologies handled within the teacher education.

regular teaching methods classes we meet on a daily basis is expressed as the type of tasks of the DO. The type of tasks of the DO of teacher education is, as it described in the introduction, to make PTs learn the disciplinary knowledge in mathematics and its teaching approaches. TEs' DOs promote PTs to learn how to construct the MO and DO of their future lessons. In this paper, we denote the didactic organization of the lessons in the teaching methods courses in pre-service teacher education by DO_{TE} . In addition, the school mathematical and didactic organizations demonstrated, or otherwise referred to during the lessons are denoted MO_{SCH} and DO_{SCH} . In the MO_{SCH} , there are types of tasks (T), which school pupils are supposed to solve, and schoolteachers' tasks (types contained in DO_{SCH}) are to support pupils to achieve the MO_{SCH} (T). These two organizations are thus intertwined with each other. The pair of the MO_{SCH} and DO_{SCH} belong to another institution (the School) than the teacher education institution, and together they form the object of the TE's praxeology DO_{TE} . (see Figure 1). We say that the didactic divide appears when the mutuality of the MO_{SCH} and the DO_{SCH} is not expressed explicitly by TEs in their DO_{TE} . TEs must thus handle this complex structure of the praxeologies: to promote the PTs to acquire the knowledge and methods necessary to construct their own future lessons where the MO_{SCH} and DO_{SCH} are interrelated.

To realize the aim of this paper, we address three research questions for a lesson in each of the three countries mentioned above:

RQ1. What are the main elements of each TE's didactic praxeologies in the lessons? In particular, (how) do they relate the didactic organisation (DO_{TE}) of each lesson to the mathematical and didactic organisation ($MO_{SCH} \leftrightarrow DO_{SCH}$) aimed for lessons concerning the determination of polygon area in school?

RQ2. What are the main differences between the three lessons, concerning research question 1?

RQ3. What institutional or social conditions and constraints can provide wider explanations for these differences?

In the next section, we present the methodology to address these research questions.

3. Methodology

In order to answer RQ1 and RQ2, we present episodes where the TEs treat a common subject matter, namely the determination of the polygon area. For each country we have selected episodes from the teaching where the TE and the PTs interact around similar aspects of this mathematical theme, to make them as comparable as possible, and at the same time represent characteristic features of the teaching in each country. We have analysed the elements (T, τ , θ , Θ) of the DO_{TE} , and the MO_{SCH} / DO_{SCH} which are presented

within the DO_{TE} of the lessons. Thereafter we highlight the characteristics of the DO_{TE} of the TEs in each country. Video recordings of the episodes were made: ‘Quantity and Measurement’ (Japan, with 53 students), ‘Area of Polygons’ (Finland, with 34 students) and ‘Area and Perimeter’ (Sweden, with 20 students). The names of the TEs are pseudonyms. The recorded lessons were transcribed in the original language, and then we translated the transcriptions into English together as we watched the videos. All analysis was made with the English transcriptions by all authors.

We should mention that unlike the Japanese and Swedish courses, the Finnish course consists of two separate sections: a lecture session and workshop session. During the lecture session, the educator mainly describes the mathematical contents, as a background for mathematics lessons in school. Then during the workshop session, which is carried out several days later, the PTs practice certain teaching scenarios meant for school pupils, but with each other in lieu of pupils. In this paper, we present one workshop session from the Finnish programme, where the PTs have opportunities to interact to each other and the TE.

The theory block of the DO_{TE} is not observable from one single lesson and therefore we asked each TE to answer some questions after they conducted their lessons. The design of this questionnaire was inspired by the Content Representation (CoRe) model [27], which was originally created as a methodological tool to develop science teachers’ pedagogical content knowledge [28]. The questionnaire consists of eight questions which may help to identify different components of the DO_{TE} and MO_{SCH}/DO_{SCH} . The data used in this analysis were from responses to the following questions:

- Q1. What do you intend the students to learn regarding this topic (area of polygons)?
- Q4. What kinds of difficulties/limitations are connected to teaching this topic?
- Q7. What teaching methods do you use to make your teaching on this topic engaging, and for what particular reasons?

Q1 is related to identifying the elements of the MO_{SCH} that were prioritized in the DO_{TE} -practice. The questions Q4 and Q7 help to identify institutional conditions of the praxeology as a whole. This is relevant to the RQ3 – investigating the wider explanations for the differences between the three countries’ lessons. To analyse the procedures of their daily lessons also ensures that the particular lessons we observed were not exceptional. The questionnaire was translated into Japanese, Finnish and Swedish, and the TEs answered it in their own languages. Then their answers were translated into English. The analysis of the TE’s answers was made jointly by the authors, using the English translations.

To support the investigation concerning the RQ3, we additionally made a small-scale comparison of each country’s national curricula, the curriculum guidelines and a few textbooks in the section concerning measurement. The curriculum is a result of the *didactic transposition* designed by different stakeholders within the education system [22]. During the process of formulating the curriculum, the original mathematical *scholarly knowledge* [22] created by the community of mathematicians, is disassembled and reconstructed into the *knowledge to be taught* [22] in a form which is more appropriate for teaching within the school systems of each country. Comparing the national curricula therefore lead us to distinguish the conditions of the construction of the DO_{TE} . We present the result of this comparison in the next section.

4. Results

4.1. *National curricula and guidelines concerning measurement*

In The Guidelines for the Japanese National Curriculum for grades 1–6 [29], the determination of length, area and volume is described in the chapter (of the Guidelines), Quantity and Measurements, positioned between the chapters of Arithmetic and Geometry. The content for each grade is described in detail with concrete teaching proposals. The Guidelines emphasize that the teaching methods are supposed to build on the pupils' previous knowledge and the pupils' various ways of solving problems. For that purpose, the Guidelines includes tables which presents the overview of the central content for grades 1–6 and for grades 7–9.

The Guidelines describes that children's learning process of measurements consists of four phases; direct comparison, indirect comparison, measurement using arbitrary objects as units, and measurement using standard units. This order is clearly followed by Japanese textbooks [30].

In the Finnish National Core Curriculum for Basic Education [31], the content regarding quantities, units and measurement for grades 1–2 is briefly described in the chapter Geometry and Measurement following the chapter Numbers and Calculations. For the grades 3–6, the content of perimeter and area is included in the chapter (of the curriculum) Geometry and Measurement following the chapter Numbers, Calculations and Algebra. In the Swedish curriculum, the content of quantities, units and measurement, perimeter and area are included in the chapter Geometry following the chapter Algebra for the grades 1–6 [32]. The descriptions of the contents consist of only a few lines. None of the two latter curricula give any practical guidelines for teaching the contents.

Unlike Japan, textbooks are not approved by a ministry in Finland and Sweden. The presentations of the contents of measurements in Swedish textbooks for grade 1 are often placed within sections covering Arithmetic (e.g. [33]), in spite of the fact that the Swedish national curriculum introduces the concepts within Geometry. In the Finnish textbooks, the concept of measurement for grade 1 is placed between the chapters of Arithmetic and Geometry (e.g. [34]). In Finland and Sweden, the four phases for the introduction of the concept of measurements are not present, unlike Japan. Some Swedish textbooks introduce direct comparison and measurement using standard units simultaneously (e.g. [33]), while some Finnish textbooks introduce the measurement using arbitrary units first, then the comparison using arbitrary objects as units, the direct comparison, indirect comparison and finally the standard units (cm) are presented (e.g. [34]). Potential tasks with an understanding of indirect comparison are not addressed in most Swedish textbooks.

The MO technologies of the four phases of measurements in the Japanese Guidelines are strongly connected to each other. For instance, the technology of using arbitrary objects as units in grade 1 is linked to the technology of area determination of rectangles in grade 4; the sum of the number of squares (which are arbitrary objects) expresses the quantity of the area. Therefore, to follow the 'correct order' of the four phases (firstly, pupils learn the direct comparison, secondly, indirect comparison, then measurement using arbitrary objects as units, and finally, standard units) is absolutely essential from the epistemological point of view. Hence, it would never happen that one introduced direct comparison and measurement using standard units at the same time or introduced measurement using

arbitrary units before direct comparison in the Japanese textbooks. One follows the order of the four phases that the Guidelines suggest.

Comparing these two contexts, we conclude that the Japanese curriculum does not give much space for different interpretations of its contents. For example, the four phases of measurement provide a suggestion for a uniform teaching approach for textbook authors and users. We assume the reason that many Swedish textbook authors position the section of measurements in the domain of arithmetic, is to enable a natural connection between area calculations and the basic arithmetical operations. This suggests that Swedish textbook authors may have different interpretations on the national curriculum, and consequently, different textbooks provide different teaching approaches in Sweden.

4.2. Lesson observation ‘quantity and measurement’ in Japan

The course *Elementary mathematics teaching methods* aims to provide the PTs with knowledge of the contents of elementary school mathematics and its teaching methods. The 12 lessons consist of the goal of mathematics education and elements of mathematics lessons, arithmetic, quantity and measurement, geometry, functions, lesson design (including problem solving) and principles for the *mathematical way of thinking*. Mr. Matsui is the lecturer of the course at a national university located in the middle part of Japan. He has worked as a mathematics teacher in lower secondary school for 14 years and as TE at the university for 12 years. This is the sixth lesson of 12 in total and it concerns the chapter on ‘Quantity and Measurement’. Episode 1 represents the first half of the lesson and episode 2 represents the second half of the lesson.

4.2.1. Episode 1: the concept of area and area determination of rectangles

Mr. Matsui explains the four phases in the process of pupils’ learning about measurement by referring to the Curriculum Guidelines and clarifies those different comparison methods for the class. Then he mentions the concept of area. In the following transcription, ‘PT’ means a prospective teacher, and ‘M’ means Mr. Matsui.

M: In grade 1, (referring to the contents overview in the Guidelines) they learn about area with direct comparison and then indirect comparison. Then it will be in grade 4 that they again learn about area. There they will compare areas using arbitrary units, and then standard units.

Mr. Matsui now demonstrates how grade 4 pupils learn the concepts of area and perimeter. He draws a rectangle (A) with grids of (6×4) and a square (B) (5×5) on the blackboard (see Figure 2).

He asks the PTs why some pupils in grade 4 think that the areas of (A) and (B) are the same. A PT answers that it depends on the sum of the width and heights, since 4 and 6, and 5 and 5 are equal; 10. Mr. Matsui remarks that most textbooks introduce the area of rectangles in this way: showing the two rectangles with same sums of perimeters and let the pupils to understand that it would not work to compare area by the perimeter. He then describes how the introduction of the standard units is usually carried out in textbooks and demonstrates a practical teaching approach:

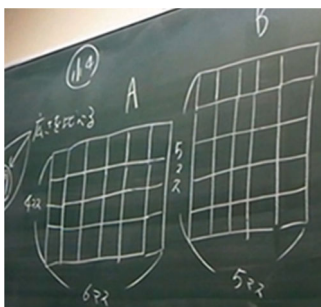


Figure 2. Mr. Matsui's figures of introduction for area determination of rectangles.

M: When they use an arbitrary units (squares in the rectangle), they count the number of the squares like this (writes down numbers 1,2,3 . . . in the grids). So counting the number of the squares is still an arbitrary measurement. Then the next stage (in the book) is to show that one square consists of the standard unit of 1 cm times 1 cm and to define it as 1 cm^2 . The introduction of the standard units concerning area is usually done in that way. Then the next stage is . . . we say 'isn't it a bit tough to count all the grids every time?' (writes 'tough') and we must encourage pupils to find out an 'easier' way to determinate (writes 'easier way'). They have already learned multiplication by grade 2, and understand that it would be determined by 6×4 and mentions that this is called 'formula' (writes 'formula'). This is supposed to be the first time pupils learn the notion of 'formula'.

4.2.2. Episode 2: area determination of parallelograms

The second half of the lesson is spent experiencing a short version of a structured problem solving approach. This approach emphasizes learners' active participation in mathematical activities, using challenging problems and collective reflections [1]. Mr. Matsui distributes to the class grid papers where a figure of a parallelogram of width 6 cm and height 4 cm is drawn, and lets the PTs find out several different methods for the determination of the area of these parallelograms which could be developed by pupils in grade 5:

M: Pupils in grade 5 have already learned direct/indirect comparison, measurement using arbitrary unit and standard units of area and the formula for area of rectangles/squares. Thus, it means that we will use all that knowledge and find out the formula for the area of a parallelogram.

Seven PTs draw pictures and explain their different solutions on the blackboard.

PT2: I moved this (pointing the right triangle on the left) here (on the right) and made a rectangle. Then the area will be 4×6 and 24 cm^2 (see Figure 3).

M: If we cut this triangle ABE and put here (the shaded section), is the area still the same? If we ask children of grade 1, they may argue that the area can be changed. This we call area preserving property. Some children in grade 1 do not understand it (writes down 'area preserving property').

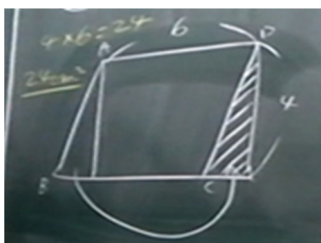


Figure 3. PT2's solution.

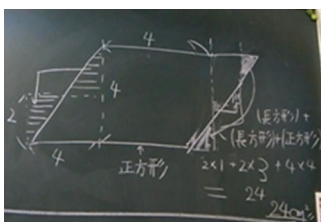


Figure 4. PT3's solution.

Since PT2 talked about moving a part of the parallelogram and making a rectangle, Mr. Matsui explains the general and crucial property of *additivity* of quantities, referring to the Guidelines.

M: Additivity is another important property. For instance, if you put 100 g play-dough and 50 g play-dough together, some younger children think that the weights will be less than 150 g. Since, what you see when you put the two doughs together is a change of shape.

When another PT explains her solution method, as shown in Figure 4, Mr. Matsui asks her:

M: You said after you made 4×4 square, you moved the top left to the bottom left, didn't you?

PT3: Here? (pointing the top left triangle)

M: Yes, there. To put this triangle to the left bottom, which kind of movement is needed?

PT3: (turning the top-triangle down) turning over?

M: Turning over? Then it sound like it was turned to up-side down.

PT4: Point symmetry.

Mr. Matsui traces the two triangles by yellow and red chalks (see Figure 5) and confirms with the class that it turns 180° . He continues:

M: When it (the red triangle) turns 180 degrees then it fits on this (yellow triangle). Point symmetry is learned in grade 6. It is not necessary to use the proper term but if the pupils have experienced this kind of activity, the lesson on point

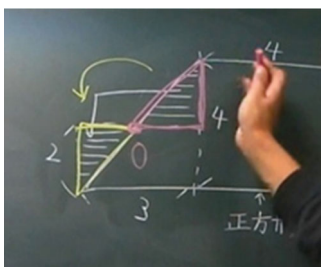


Figure 5. Mr. Matsui's explanation of point symmetry.

symmetry in grade 6 will be richer. In addition, rotational translation will be learned in grade 7. The future lesson will be more meaningful if you consciously apply and illuminate these related topics.

Thereafter, he compares the different kinds of *shifts* between PT3's *rotation* and PT2's *parallel translation*. Finally, he explains the formula for the area of parallelogram as height times length since the geometric transformations shows that the height and length of parallelograms corresponds to those of rectangles. In the same way, he gives a final task to determine the area of a trapezoid, using same didactical approach. Some of the PTs transformed trapezoids to a double size of the original parallelogram and made a rectangle. From this solution, the class concluded the formula for the area of a trapezoid to be $(a + b)h/2$.

4.2.3. Analysis of the episodes

The type of tasks of the DO_{TE} in episode 1 aim to help the PTs learn how to construct the praxeology of 'making the formula for area of rectangles'. Mr. Matsui's DO_{TE} includes several techniques, such as mentioning a typical teaching approach presented in textbooks. Yet, the most crucial technique of his DO_{TE} is to refer to the Guidelines. He describes the teaching/learning process from the direct comparison to the area of rectangles by referring to the contents overview. In so doing, he exemplifies the four phases of the measurement by a case, the area of rectangles. When he explains the process of establishing the formula, the specific terms from the Guidelines are described: direct/indirect comparison, arbitrary unit, additivity, which are components of the MO_{SCH} technology. School pupils do not have to master applying these terms, but the PTs do, in order to understand the whole construction of the MO_{SCH} better. The DO_{TE} technique of discussing the use of the different terms makes the technology of the MO_{SCH}/DO_{SCH} explicit. The main tasks of the DO_{TE} in episode 2 are: 1. helping the PTs learn the MO_{SCH}/DO_{SCH} of 'determination of area of a parallelogram and trapezoid', 2. letting them anticipate pupils' solution methods on this topic and examine the viability of such methods.

Mr. Matsui lets the PTs participate in a short version of an example lesson using the structured problem solving approach. This is a main technique of the DO_{TE} in this episode. Mr. Matsui lets the PTs follow up one of the most important techniques of the DO_{SCH} – whole-class discussions. The whole-class discussions lead to the discourse of several mathematical techniques and this in turn leads to the use and establishment of a richer technology and theory of the MO_{SCH}. It means, through discussing/comparing the various

solving methods, pupils recognize that the rigid transformation is a fundamental concept in order to reach an algebraic interpretation of area determination. In the episode, Mr. Matsui asks PT3: ‘which kind of movement is needed?’ This is a question to make the technology of the MO_{SCH} technique explicit: letting the PTs realize why such and such technique can be used.

These components of the DO_{SCH} promote the construction of a praxeology where the knowledge from the previous grades to the forthcoming grades are connected and re-established. Mr. Matsui refers to the Guidelines and describes how to make grade 6 lessons richer by, e.g. discussing the notion of rotation in the grade 5 lesson. This is a direct technique of the DO_{TE} , which support the PTs to grasp the DO_{SCH} technology– applying the statement of pupils’ previous experienced local MO_{SCH} .

4.2.4. Analysis of the theory block of the DO_{TE}

Regarding Q1, ‘What do you intend the students to learn regarding this topic (the making of formulas for area determination)?’ Mr. Matsui answered: ‘Areas of polygons can be determined in various ways by using pupils’ previous knowledge’. Also, he stressed that the PTs should be able to apply certain didactic terms: ‘The terms describe the various methods of area determination and help the PTs in understanding the pattern of the different solving methods’. Regarding Q7, ‘What teaching methods do you use to make your teaching on this topic engaging, and for what particular reasons?’ he emphasized ‘to consider having the pupils’ perspective’, ‘confirming the previously learned items’, ‘to consider having various solution methods’. He also remarked that it is important to let the PTs know intimately the flow of a lesson with a problem solving approach, which are: reason individually → discuss with neighbours → present the solutions in class → respond to comments from the lecturer. He described how he treats the simulated whole-class discussion with the PTs: he asks some of the PTs who use typical solution methods, to present and explain them to the class. During the presentations, he usually instructs them not directly but by his gestures, where/how they should stand by the blackboard, if the volume of their voice and speaking tempo are appropriate, etc.

These answers indicate, in line with the lesson observations, that the focus of his DO_{TE} is on making the PTs learn how to relate the local MO_{SCH} s of individual lessons on a larger time-scale and thus to construct a complex MO_{SCH} .

Mr. Matsui’s remarks about the importance of using the specific didactic terms and of knowing the flow of the structured problem lessons, indicate that these statements are crucial components of the theory level of the DO_{TE} , which are shared to a large extent within the teacher education. The purpose is to make the theory block of the DO_{SCH} explicit. Since the importance of using the specific terms and applying the problem solving are clearly stressed in the Guidelines, to stress these two issues for the PTs is indispensable. One of the authors have attended method courses in several other universities in different regions in Japan, and observed that every TE refers to the Guidelines and explains the didactic terms described there.

4.3. Lesson observation ‘geometry’ in Finland

The course *Didactics of Mathematics* for PTs for grades 1–6 in a state university located in southern Finland provides knowledge of the contents of elementary school mathematics

and its teaching methods to promote children's learning of mathematics. As described in the methodology section, the course is carried out with lecture respective workshop sessions. The 12 lecture sessions treat basic arithmetic, numbers, fraction and decimals, percentages, units and quantity, geometry, probability, inductive way of working, problem solving, curriculum and observation of pupils' way of thinking. The lecturer, Ms. Ahonen, has worked as a mathematics teacher in primary and lower secondary school for 5 years, thereafter as a TE for 16 years.

In the lecture session, 'lesson 8, Geometry', given several days before, the following concepts were explained: classification of geometrical figures, line symmetry and rotational symmetry, perimeter and the area of polygons, properties of the circle and concept of scale. In the workshop session discussed here, the PTs move between six different tables to work practically with the above-mentioned concepts. The PTs work in groups using a compendium (work sheets with descriptions) written by Ms. Ahonen. The compendium gives instructions on the target knowledge of related mathematical concepts of each table and its teaching methods including some tasks for school pupils. Ms. Ahonen moves between the tables to give advice to the PTs on how to solve the tasks the compendium suggests. Here we present the episodes from the workshop of 'area of polygons' and 'area and perimeter', since the topics treated in these workshops are highly relevant to the topic, which was dealt with in the Japanese lesson.

4.3.1. Episode 1: 'area of polygons'

The description of 'area of polygons' in the compendium starts with the following:

Area of polygons is learned in grades 5-6. The formula regarding area determination should be treated inductively. That is, by looking at few particular cases, one derives the general rules together with pupils.

In accordance with the description in the compendium, one PT in a group plays the 'teacher role'. The 'teacher' explains how to determine the area of rectangles by using grid paper with squares of 1cm^2 .

PT5: How many squares are there now? (points at rectangle of 5 grids in length and 3 in heights)

PT6: We have 15 squares.

PT5: We look at this one (she draws another rectangle of 3 squares in length and 2 in height)

PT5: You can count this here (points at the length 3) and here (the height 2), 3 times 2. When we multiply the width by the height, we get the area (of the rectangles).

Ms. Ahonen ('A' in the transcript) has been watching this group and remarks:

A: Here, we see that the different phases of how one teach the formulas using the inductive way of learning. It means, in reality, there are several cases. Thousands of different cases (of different rectangles) from those you have done here. After you have verified the formula, you can begin to apply this formula,

namely, letting pupils work with tasks from textbooks. So they practise applying the formula. Afterwards it is good to summarise what you have taught and ask yourselves: in which case can you apply this formula? For example, this method (formula) is not suitable for triangles.

The next task is to find out the formula for the area of a parallelogram. The compendium describes the method of parallel translation (however, the term parallel translation is not used in the lesson) and explains that one can use the same formula as for rectangles. PT6 reads aloud the text:

PT6: Pupils draw various parallelograms on the paper. By cutting, they will find out how to form parallelogram to a rectangle.

The PTs cut papers and transform the figure to make rectangles. PT6 continues reading aloud the heading '*The limits of the formula and special cases*' in the compendium to her peers:

PT6: 'One cannot transform a trapezoid back to a rectangle'.

They do not discuss the exact significance of 'the limits of the formula' such as, why One cannot transform a trapezoid back to a rectangle, and in what way then one can teach a method of calculating the area of a trapezoid – they move to the next task: Area of Triangles. They explain to each other the method of area determination by reflecting the instruction of the compendium:

Make a parallelogram by drawing two similar triangles and let pupils notice that the area of one of the triangles is half the area of the parallelogram

Additionally, the compendium describes the property of an area of right triangles under the heading 'The limits of the formula and special cases' as:

Right triangles are a special case where one can determine the area by its cathetus

When PT5 has read these descriptions, she wonders if one can use a rectangle and divide it into two right triangles, instead of using a parallelogram as the compendium suggests. She draws a diagonal in a rectangle and asks Ms. Ahonen:

PT5: Which is the smartest way to determine a triangle's area, starting from a parallelogram or a rectangle?

A: (Points at the rectangle PT 5 made). But the thing is that all triangles do not have right-angles.

PT5: Ok ...

A: But it is good that pupils verify different ways that the area of triangle is defined by 'Base times height divided by two'.

4.3.2. Episode 2: 'area and perimeter'

The task at this table is to make different kinds of quadrangles having area 12 cm^2 using a Geo-board. Ms. Ahonen encourages the PTs to make even irregular quadrangles with

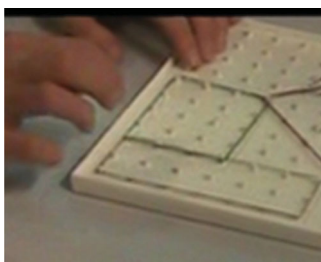


Figure 6. The different perimeters on Geo-board.

the same area. The PTs try making several different shapes of quadrangles and eventually notice that the perimeters do not need to be the same even if the areas are the same. Ms. Ahonen then asks the group:

- A: In which way do the forms of the figures influence the perimeters? What does a figure look like in order to have big perimeter?
- PT7: Like this (makes a long slim rectangle) (see Figure 6).
- PT8: Why does it work in that way? Are there any rules?
- A: It has to do with the inductive way of working in lower grades.

We can derive understanding toward this phenomenon through many single cases in the lower grades. That is good enough on these levels (lower grades).

4.3.3. Analysis of the episodes

In episode 1, the first task of the DOTE is to help the PTs know the ‘inductive way of learning’ which promotes pupils to find out the formula of area autonomously. The second task is to let the PTs learn a specific model of MO_{SCH}/DO_{SCH} of the area determination. There, the techniques of the DO_{TE} are using the compendium with exercises and using role-play. The compendium describes directly a part of the MO_{SCH}/DO_{SCH} . For instance, the MO_{SCH} for finding out the formula is, using figures, counting of the grids, and the multiplication. The statement of (not mathematical) induction is the most evident technology of the MO_{SCH} for justifying these techniques. The other MO_{SCH} technologies are standard units and commutative property of multiplication. Consequently, the technique of the DO_{SCH} , as the compendium suggests, is to let pupils try to count the number of the squares of different rectangles to find out the formula by themselves.

The compendium suggests that one should apply pupils’ previous knowledge to establish ways to compute the area of polygons: from the area of rectangles to parallelograms and then finally that of triangles. However, unlike the Japanese case of finding the formulae of parallelogram and trapezoid, the PTs in this workshop did not have an opportunity to discuss the validity of the formula: in the episode with PT6, who was reading aloud the description in the compendium One cannot transform a trapezoid back to a rectangle, PTs in this group did not have any discussion about why the method of area computation for parallelograms would not work for trapezoids. Also, in the episode with PT5, concerning the area of triangles, Ms. Ahonen does not discuss this epistemological connection. She remarks that ‘not all triangles have right-angles’ but does not emphasize the importance of

using a systematic approach in the MO_{SCH} to make PT5 realize why it is (according to the compendium) advisable to work with parallelograms and not rectangles.

These episodes indicate a limitation of the workshop: Even though the PTs are interested in learning more about the theoretic level of the MOs described in the compendium, the workshop lacks opportunities for discussions and institutionalization of the theory block of the MO_{SCH}/DO_{SCH} , since the TE's primary focus was on emulating the praxis block of the MO_{SCH}/DO_{SCH} . A similar phenomenon is observed in the dialogue concerning the perimeter of a quadrangle in episode 2. PT 8 wants to know more about the theory and technology regarding the area and perimeter computations. However, Ms. Ahonen's DO_{TE} techniques consistently aims to inform the PTs about the inductive way of learning, where an understanding of a phenomenon is absorbed from many single cases. It does not aim to construct a deeper technology of the DO_{SCH}/MO_{SCH} . On the other hand, we have not had an opportunity to observe Ms. Ahonen's corresponding lecture session (implemented some weeks ahead of the workshop), where theoretical (didactical and mathematical) perspectives on the DO_{SCH}/MO_{SCH} concerned by the workshop lessen had presented to the PTs. This might explain her focus on the praxis block of the MO_{SCH}/DO_{SCH} during the workshop.

4.3.4. Analysis of the theory block of the DO_{TE}

According to her responses to the questionnaire, Ms. Ahonen's intention, for the PTs learning regarding the determination of the area of polygons, is a hierarchical structure: the area of triangles is based on the area of parallelograms which in turn is based on the area of rectangles. Ms. Ahonen intends to give the PTs 'a teaching model' concerning the formula for area determination. She considers this section a good starting point for the PTs to learn the inductive rule for teaching, which is important for all mathematics teaching. In response to Q4: 'What kinds of difficulties/limitations are connected to teaching this topic?' she describes the PTs' fragmental knowledge about the formulas for area determination. They can apply the formulas but lack a deeper interpretation of why they work. During her lesson, she often discusses pupils' misconceptions of area and perimeter to let the PTs realize their own misconceptions. Regarding Q7 about the teaching procedures, she describes the combination of lectures, homework following the implementing of workshops with manipulatives and group discussion. She remarks, 'I attempt to emphasize those items which the PTs have difficulty with during my lecture. Some learn by doing and others learn by discussions with groupmates.'

Ms. Ahonen's answers above indicate that the application of the inductive way of working is a crucial component of the theory block of both the MO_{SCH} and DO_{SCH} , since she remarks that the inductive rule for teaching is important in all of the mathematics teaching. At the same time, we can state that the maxim of 'using an inductive way of learning within teacher education' is part of the theory (Θ) of the DO_{TE} that justifies the praxis block of the DO_{TE} . This statement is considered a crucial component of the course. Her remark that she provides a 'teaching model' for the PTs by using the compendium and workshops indicates that the theory of the DO_{TE} for the second task 'to let the PTs learn a specific model of MO_{SCH}/DO_{SCH} ' is a traditional statement of learning by practicing: the consideration that the PTs will learn the praxis of the DO_{SCH} by following the compendium and doing the role-play.

Neither during the observation nor in the questionnaire does Ms. Ahonen's DO_{TE} indicate that she wants to mediate any specific theory of the DO_{SCH} besides the inductive way. She states her concern about the PTs' fragmental knowledge of mathematics, but does not remark on what kind of mathematics the PTs are supposed to learn.

4.4. Lesson observation 'area and perimeter' in Sweden

The course *Mathematics and Learning for Primary School, Grades 4–6 Teachers II, Geometry*, in a state university located in the middle of Sweden, treats mathematical knowledge in geometry and mathematical education in relation to the current Swedish curriculum. Examples of content covered are: An historical perspective of geometry, mathematical terminology within geometry, analysis of pupils' knowledge in geometry, a didactical approach to teaching geometry from theoretic perspectives, different forms of representation in geometry and the importance of using mathematical expressions. The lecturer Ms. Nilsson has worked as a mathematics teacher in grades 4–7 for 13 years and as TE for 12 years. The subject of today's lesson is the concept of area and perimeter and area determination. We present here two episodes from the lesson, which lasted 150 min in total.

4.4.1. Episode 1: the concept of area

Ms. Nilsson ('N' in the transcript) starts the lesson with the definition of the polygons and lets her PTs consider their own interpretations of area and perimeters.

N: When you teach about new concepts, it is better if you reflect by yourself first. What do I know about this? So it will be a good starting point.

Thereafter, Ms. Nilsson gives the PTs five group-exercises concerning area and perimeter. The first exercise is to measure the area of the rectangular chair sheets using a covering by a grid of ice cream sticks. In this exercise, Ms. Nilsson lets the PTs discover the concept of the area by using arbitrary units and the formula for area determination.

N: It is very common that one starts with the formula when one learns a new concept. One might not understand where the formula actually comes from. You see the rectangular figure here; the area is the number of the squares on the one side (points at one side) multiplied to the number of squares on the other side (points at the other side). Then it will be a region, which is covered by x numbers of squares. So, if one counts the number of all squares, which can be quite many, then one may discover that it will be easier if one multiplies one side with the other side; we can say the length times width.

Thereafter, Ms. Nilsson shows the class statistical data from TIMSS 2007 for grade 4 and 8 about Swedish pupils' misconceptions on area determination. She concludes by referring to the Swedish national curriculum:

N: When we look at the curriculum. (Shows the text from the national curriculum on a slide) Here you see about grades 1–3. It says almost the same thing also for

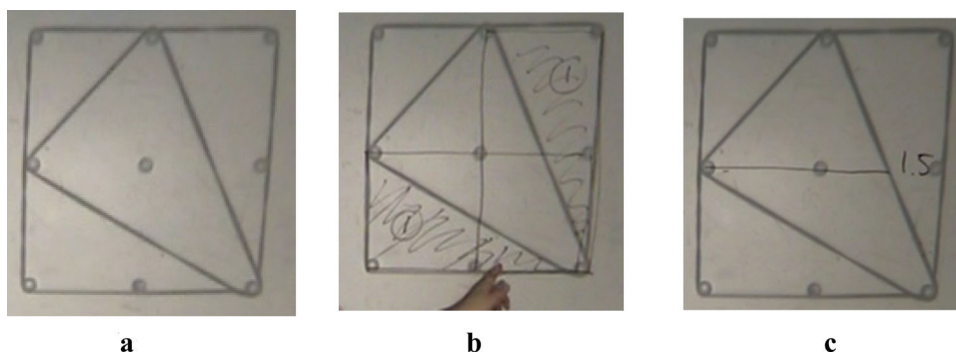


Figure 7. (a) an isosceles triangle (b) with an auxiliary line (c) PT 10's figure.

the grades 4–6. I will not read it out aloud, but you can see what the emphasis in the text is about.

4.4.2. Episode 2: area determination of polygons using Geo-board

The sixth exercise is to determine the area of geometrical figures by using a *Geo-board*. Ms. Nilsson demonstrates a method for area-determination of an isosceles triangle using a rubber band outlining the triangle. She divides the framing rectangle (a square) into four squares. Thus the sides of the triangle occur as diagonals of three rectangles within the frame. Now the PTs ponder the method for area-determination of another isosceles triangle in groups (see Figure 7a).

N: Think about the diagonal. The diagonal must go from the one corner to the other. Not the half way. (PT 9 raises his hand) Yes?

PT9: I use . . . the diagonal to determine the under triangle.

N: Ok. This part (the rectangle on the under part) is 2. 2 divided by 2 is 1 (writes $2/2 = 1$). What did we do here? We circumscribed the whole and made a rectangle. And the rectangle has 4 area units. Then we begin to take away this (triangle) part. We take away one. (writes $4 - 1$). Then we have a new rectangle here. And in the same way: $2/2$ is 1. I take away this part as well. So you see? We have already taken away those (points at the whole right triangles on the bottom and on the right). We do not take away those now (pointing at the small right triangles on the bottom and on the right) (see Figure 7b).

In the middle of Ms. Nilsson's description, PT 10 suddenly wonders if he can use another method:

PT10: I use this as the base, which is 1,5 (see Figure 7c). And determine the area of the two (upper and under) triangles and add them. The base is 1,5 and the heights are both 1. And I divide it by 2 (writes $1,5/2$). It is 0,75. So I add them. Then it is 1,5.

N: Thank you, (to the class) does it make sense?

PTs: Yes (some).

PTs: No, it does not (some others).

N: I can say, one (meaning ‘pupil’) can understand this method if one is more skilled in mathematics.

Then PT11 wanted to present some other method at the whiteboard. However, he could not completely explain his solution process to the class. Ms. Nilsson comments to PT11:

N: It is good that you have your own knowledge. We should have it. But we must start with something basic when we work with children, so that we do not lose them in the process.

PT12: Can you explain your method again?

N: The one I started with? Yes, let’s finish this. We start from the beginning.

Ms. Nilsson did not let PT11 complete presenting his *different* solution method to the class. However, she explains willingly *her* method again when PT12 asked her to do it.

4.4.3. Analysis of the episodes

The tasks of the DO_{TE} in episode 1 are: firstly, to encourage the PTs to establish their own (correct) perceptions of area and perimeter. Secondly, to inform the PTs of the technique in the MO_{SCH} of finding the formula for the area of rectangles by covering with squares. The technique to realize the first task is to let the PTs write down their current perception of the concept of area. Ms. Nilsson’s intention is that the PTs will validate their actual perception of the concept during and after the exercises.

The second task deals with exactly the same issue as that described in the Japanese and Finnish lessons – let pupils find the formula by counting the number of the squares in rectangles. Ms. Nilsson’s DO_{TE} technique is to describe the MO_{SCH} technique directly for the PTs: ‘it will be the region, which is covered by x numbers of squares . . . one count the number of all squares . . . one multiplies one side with the other side’. This technique does not promote the PTs understanding of the MO_{SCH} technology that justifies the valid MO_{SCH} technique. Neither does it illustrate the DO_{SCH} technique to use for teaching the MO_{SCH} technique to pupils.

In the second episode, the task of the DO_{TE} is to have the PTs experience a model of specific MO_{SCH}/DO_{SCH} of area determination using Geo-board. This technique of DO_{TE} – let the PTs experience a lesson ‘Geo-board with group discussions’ – generated more mathematical techniques than Mrs. Nilsson had expected. The sides of a rubber-band polygon on a geoboard occur as diagonals on rectangles with integer coordinates. Mrs. Nilsson’s intention was to train the PTs’ algorithmic skills with one technique based on the technological observation that the diagonals halves of the areas of these rectangles. As the technique by PT10 suggested, one can make other observations using the integer coordinates of the vertices. She let PT10 explain his alternative technique, but did not validate it by, say, verifying that the base is 1.5 length units as stated, e.g. using the similarity of triangles. She commented to the PT11 ‘we must start with something basic when we work with children’ when he wanted to explain his method. Her intention was not to discuss the viability of different mathematical techniques for grade 5 but to establish a certain MO_{SCH} technique which is possible for all PTs to manage.

4.4.4. Analysis of the theory block of the DO_{TE}

In her response to the first question (TE's intention for the PTs to learn on the area determination), she firstly states that the PTs should be aware of their own perceptions on the concept of area and perimeter: 'That they understand the concepts and methods by themselves is a prerequisite. They must be able to teach to give pupils understanding, in order to create interest and commitment in the classroom'. She strongly emphasizes her PTs' difficulties and limitations concerning geometry. Some of them have learnt the formulas for area determination superficially and sometimes incorrectly. Also, the PTs' perception that 'geometry is a difficult subject' blocks their learning process. Furthermore, the PTs have not developed mathematical terminology allowing them to explain their solutions properly. To deal with these difficulties, she uses manipulatives to give them concrete ideas about different mathematical concepts and train them to establish their own interpretation of the concepts. In order to enhance their mathematical communication skills, she uses group discussions with workshops.

Hemmi and Ryve's research [17] suggests that the 'Swedish discourse on classroom teaching builds on a rather extreme interpretation of constructivism' (p.516). Ms. Nilsson's remark that her first didactic task is to make the PTs be aware of their own perceptions of the concept the area and perimeter indicates that a constructivist theory of learning underlies the justification of her DO_{TE} technique. Further, her strong concern about the PTs' anxieties regarding learning geometry and her attempt to *nourish* the PTs' interest toward geometry, point to the influence from a psychological view of teaching, focusing on the development of students' self-efficacy [35].

Ms. Nilsson's didactic technique for supporting her goal that PTs become 'able to teach so as to give pupils an understanding in order to create their interest' is to demonstrate an 'ideal' lesson example. The DO_{TE} theory that justifies this praxis is a traditional statement of learning by practicing: one acquires a method by watching a demonstrated teaching approach.

5. Discussion

The overall task of the DO_{TE} of the three educators is more or less common: to help their PTs learn to construct the MO_{SCH} and DO_{SCH}. However, the three TEs' techniques for realizing their aim are quite different. We summarize here the results and compare the TEs' main DO_{TE} (see Table 1) to give the answer to the RQ1: *What are the main elements of each TE's DO_{TE} in their lessons? How do they relate the DO_{TE} to the MO_{SCH}/DO_{SCH}?* Also, RQ2: *What are the main differences, concerning the RQ1?*

A significant characteristic of the DO_{TE} technique of the Japanese TE which differs from the Finnish and the Swedish is the *theorizing* of the MO_{SCH}/DO_{SCH} by using *several technical terms* explained in the Guidelines. Regarding the formula for the area of rectangles/parallelograms, the Japanese TE uses the specific terms *direct/indirect comparison*, *arbitrary unit*, *additivity* to make the theory block of the MO_{SCH}/DO_{SCH} explicit, while the Finnish TE used the general didactic term *the inductive way of learning*, and the Swedish TE described the MO_{SCH} technique for the PTs without demonstrating the DO_{SCH} technique to achieve this MO_{SCH}. According to Iwasaki and Miyakawa's study [36] of the process of Japanese teachers' development, the teachers begin to use the technical terms at quite early

Table 1. Task, technique and the theory in three TEs' DO_{TE}.

	DO _{TE} Task (type) (T)	DO _{TE} Technique (τ)	DO _{TE} Theory (Θ)
Japan Academic/ theoretical model	T(a) To help the PTs learn a model of the MO _{SCH} /DO _{SCH}	τ (a) To address some solution methods and related issues in specific terms by studying the Guidelines	Θ (a, b) The shared statement: the specific didactic terms described in the Guidelines should be taught.
	T(b) To illustrate how to link the previously experienced MOs	τ (b) To refer to the Guidelines and textbooks	Θ (c) The shared statement: The structured problem solving approach should be introduced to PTs
	T(c) To anticipate pupils' way of solving problems, and examine the viability of the different solutions	τ (c) To demonstrate the structured problem solving as emulated in a 'short' lesson	
Finland Rehearsal model	T(d) To help the PTs experience 'inductive way of learning'	τ (d, e) To let the PTs follow the compendium by emulating the teaching approach applying a 'role play'	Θ (d) The inductive way of learning should be taught within the method course
	T(e) To help the PTs learn the suggested model of specific MO _{SCH} /DO _{SCH}		Θ (e) The educational principle of learning by practicing: by emulating a specific teaching method one acquires the method
Sweden Immersion model	T(f) To help the PTs experience a model of specific MO _{SCH} /DO _{SCH} of area determination	τ (f) To let the PTs experience a short version of the lesson and teach them a specific τ of the MO _{SCH}	Θ (f) The educational principle of learning to teach by experiencing a teaching model as a pupil: by experiencing a suggested teaching approach, one acquires the method.
	T(g) To help the PTs to establish their own (correct) perceptions of area and perimeter and validate its propriety	τ (g) To let the PTs write down their current perception of the concept of area	Θ (g) Constructivist theories
	T(i) To nourish the PTs' interest and skills in geometry	τ (i) Group-work, workshops	Θ (i) The theory of self-efficacy

stages in their career: 'These terms principally allow teachers to draw attention to significant facts – the nature of mathematical problems, teachers' acts, students' acts, etc. – in the complicated teaching and learning situation, and apply some labels to them' (p.91). Consequently, the use of this language makes it explicit for the PTs how the MO_{SCH} and the DO_{SCH} are *mutually connected*. In the Finnish case, the existence of an explicit technology of the DO_{SCH} (the inductive way of learning) indicates that the Finnish DO_{TE} also aims to theorize the MO_{SCH}/DO_{SCH} to some extent. It gives a particular method for constructing the practice block of the MO_{SCH}/DO_{SCH}, but without much focus on illuminating the mutuality of the MO_{SCH} and DO_{SCH}. Neither, does it demonstrate how to construct a sequence of epistemologically connected MO_{SCH}/DO_{SCH} in the long term. In the Swedish case, the practice block of the DO_{TE} is *individually* designed by the TE, since a collectively shared and generally adapted theory block of the DO_{TE} is absent. If a different TE would be in charge of this course, the structure of the lessons could be quite different even at the same university.

A similar technique shared between the Japanese and the Swedish DO_{TE}, which differs from Finland, is that of letting the PTs participate in a short version of an emulated lesson using problem solving. Both TEs aim to demonstrate a model of MO_{SCH}/DO_{SCH} for the PTs, and immerse them in it. The Japanese structured problem solving establishes

a complex MO_{SCH}/DO_{SCH} , by using well-constructed initial problems and the following whole-class discussions, while the Swedish correspondent contains a single task, since the Swedish TE's main focus is to train the PTs' algorithmic skills. Consequently, the Swedish TE's performance is more that of a schoolteacher rather than a TE, and thus the boundary between the DO_{TE} and the DO_{SCH} becomes unclear in the Swedish lesson.

Considering RQ3: *the institutional explanations for those differences*, we describe the notion of *paradidactic infrastructure* [21]. The paradidactic infrastructure is conditions that affect the teaching related practice outside the classroom praxeology. Japanese *lesson study* is a typical example of such a practice, since it is teachers' 'goal oriented long term collaboration beyond the classroom' [37,p.187]. Within the process of lesson study, Japanese teachers need specific terminologies to communicate with each other, and the most of these terms are clearly described in the Guidelines. Usually, most Japanese TEs participate as advisers/commentators for teachers during 'open lessons' [38] which often are held as a part of lesson study. Thus, teaching and adopting a common set of terms is a crucial component within the method courses in Japan. In the observations made for this paper, neither the Finnish nor the Swedish educators discussed the national school curricula to any significant extent. As was mentioned in the previous sections, both the Finnish and the Swedish national curricula describe the MO_{SCH} and DO_{SCH} in broader and less specific ways than the Japanese counterpart. These differences are also reflected in responses to the TE questionnaire. The Japanese TE focuses on very specific mathematical and didactical aims of his lesson, and how they relate to the school curriculum. The Swedish and Finnish TEs give broader aims, such as filling gaps in PT's mathematical knowledge, recognizing and overcoming their own mathematical misconceptions (Finland), and promoting students' self-efficacy (Sweden).

5.1. The types of DO_{TE}

The Japanese method courses provide established theories that are adopted nationally by universities and mathematics teachers. One reason behind this is the well-maintained paradidactic infrastructure shared by the community of the TEs. In this case, the Japanese model is the most 'university-like' and therefore can be characterized as theoretical or academic.

In the Finnish lesson, the compendium gives a specific set of techniques both for the MO_{SCH} and DO_{SCH} . This DO_{SCH} is then enacted when the PTs *perform as teachers* in the role-play in front of their PT-colleagues. They rehearse the proposed model during the workshop, as a kind of preparation to teach in real classrooms. This technique is justified by the classic educational principle 'learning by practicing'. Hence, we denote the Finnish case as a rehearsal model. However, several different explanations can be given from the viewpoint of the paradidactic infrastructure in Finland: as in Japan, the school-based teacher education [18] almost takes place within the teacher education institution, due to the cooperation with so called university practice schools [18,p.137–140]. Thus, the Finnish PTs have other opportunities to rehearse instances of DO_{SCH} in these schools. Secondly, active Finnish mathematics teachers frequently use a *Teacher's Guide* and its structure and main content are quite similar between different publishers [39]. Thus, those teacher's guides function as a crucial provider of the praxis part of the DO_{SCH} for Finnish teachers in service. Thirdly, the teaching traditions, like applying a balanced combination of lectures and

homework [17] are shared within the community of TEs. In that sense, it is predetermined for the Finnish TE what are the crucial components to be taught within Finnish teacher education.

Unlike Finland and Japan, university practice schools do not exist in Sweden, and the Swedish teachers in service do not generally use a teacher's guide for designing their lessons [40]. It can be stated that a paradidactic infrastructure is not explicitly shared by Swedish teachers. The Swedish TE makes the PTs experience the MO_{SCH}/DO_{SCH} without theoretical explanation and *immerses* them into the MO_{SCH}/DO_{SCH} techniques demonstrated during the simulated short version of the lesson. The Swedish PTs have no opportunities to rehearse a model DO_{SCH} -practice during their university based course. Instead, they experience something like acting as school pupils during Ms. Nilsson's model lesson. For that reason, we call the format found in the Swedish case an immersion model for a methods course. A main difference between this model and the Finish rehearsal model is the role which the PTs get to practice (pupils in the former and teacher in the latter).

6. Concluding remarks

In this paper, we have investigated the didactic praxeologies (DO_{TE}) realized in mathematics teacher education lessons in Japan, Finland and Sweden. In the Japanese lesson, the focus of the DO_{TE} is to convey and exemplify theoretical blocks of MO_{SCH} and DO_{SCH} which are to a large extent prescribed by the national curriculum. The theoretical content of the lesson is supported by the use of well-established technical terms to describe school mathematics and related *didactic phenomena*, and didactic theories such as structured problem solving which are widely shared by Japanese teachers.

The Finnish DO_{TE} is based on a prior lecture on the inductive way of learning mathematics, and lets the PTs practice 'inductive teaching' techniques (DO_{SCH}) by a kind of role-play where students act as both pupils and teachers, following given lesson scripts (compendium). In the Swedish case, the TE immerses the PTs in the demonstrated MO_{SCH} based on principles informally inspired by psychological ideas such as self-efficacy and constructivism. We contend that the presence, in Japan, of a rich, shared, documented and content-specific theory of MO_{SCH} and DO_{SCH} , makes it possible for the Japanese TE to engage in a relatively classical university model of teaching, in which these theories are taught directly, and only exemplified.

In both Finland and Sweden, the corresponding theories remain very general and difficult to relate to actual teaching tasks. However, in Finland, there is also a rich, shared and documented praxis level of DO_{SCH} , within teaching guides and teacher education compendiums; this leads to the model of rehearsing those practices. In Sweden, the teacher educator simply demonstrates, with the teacher students as 'pupils', what she considers good DO_{SCH} -practice. Thus, in all three countries, we find strong explanations for the different choices of DO_{TE} in the different paradidactic infrastructures and resources for mathematics teaching which are available in each country. Certainly, the empirical data of this study is very limited, but the alignments between the striking differences in DO_{TE} and similarly strong differences in the conditions and constraints of DO_{SCH} in the three countries, lend support to our hypothesis that the differences found are far from coincidental, and reveal deeper and more general differences in the

ways in which mathematics teacher education is done and conceived of in the three countries.

Note

1. Bergsten and Grevholm note that the term ‘pedagogical knowledge’ they use, includes Shulman’s [15] notion of pedagogical content knowledge, curriculum knowledge (knowledge of how to sequence topics and use materials in teaching) and knowledge of general issues in education.

Disclosure statement

No potential conflict of interest was reported by the authors.

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